On Positive Relational Calculi

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Abstract

We discuss the question of inclusions between positive relational terms and some of its aspects, using the form of a dialogue. Two possible approaches to the problem are emphasized: natural deduction and graph manipulations. Both provide sound and complete calculi for proving the valid inclusions, supporting nice strategies to obtain proofs in normal form, but the latter appears to present several advantages, which are discussed.[1]

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1 Introduction

The positive relational language corresponds to the fragment of the De Morgan-Peirce-Schröder-Tarski relational calculus [10] without complementation or empty relation. Despite its simplicity and limited expressive power, it has interesting applications, for instance to relational semantics of (possibly non-deterministic) programs. In this paper, we examine the question of valid inclusions between positive relational terms and some of its aspects. Two possible approaches to the problem are emphasized: natural deduction and graph manipulations. Both provide sound and complete calculi for proving the valid inclusions, supporting nice strategies to obtain proofs in normal form. The graph relational calculus appears to present several advantages, being powerful, simple and playful.

The structure of this paper is as follows. In the remainder of this section we describe the positive relational calculus and state the problem of establishing its valid inclusions. In Section 2, we briefly examine a first approach to it: an equational calculus. In Section 3, we study an alternative approach leading to a natural deduction

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system. We formulate the rules and prove they are sound and complete. We also present a strategy for finding proofs using the natural deduction rules. Then, in Section 4, we introduce a third approach: a graph calculus. This is a refinement of the proof strategy of the natural deduction system into a simple and playful formal system. In presenting the graph calculus, we concentrate mainly on its intuitive aspects. Historical and technical details are presented elsewhere [12, 13], where a list of references can be found. Finally, Section 5 discusses some ongoing work and perspectives.

Here, we will use the form of a dialogue to present the issues involved. Two logicians, *Asinus* (**A**) and *Buridanus* (**B**), debate the usage of graphs to prove the valid inclusions of the positive relational calculus. Asinus, a very open-minded logician, tries to convince Buridanus that employing graphs is good, whereas Buridanus, a more conservative one, is reluctant to accept Asinus's somewhat unorthodox ideas.

Asinus and Buridanus meet. Buridanus explains to Asinus a problem he is working on.

A – Howdy, dear master Buridanus. Long time, no see...
B – Good morrow, Asinus. Indeed, I have been absent from the Agora Logicorum for quite some time.
A – Oh, what have you been up to?
B – I have been working on an interesting problem.
A – Really? What is it? Pray, do tell me.
B – It is the problem of deciding the valid inclusions between positive relational terms. It all started with my quadrivium class. Taking the usual relational concepts, except for complementation and empty relation, you get the positive relational terms.
A – I imagine that you would lose some expressive power with such a restriction of relational notions to the positive case, wouldn't you?
B – Indeed, there is some loss, but using inclusions between these positive terms you can express some interesting properties of relations. For instance, relational methods have been used for program semantics [6, 2], since it is quite natural to view the behavior of a (non-deterministic) program as a set of input-output pairs.
A – Under this viewpoint, it is reasonable to compose programs and, in the non-deterministic case, to unite them. But, you did say "relational methods"; don't you need complementation in programs?
B – Complements may appear in tests. But, a test like **if** $x = 0$ **then** S **else** T **fi** uses the complement of a set (of states), rather than of a relation (between states).
A – Yes, in a liberal view, a program is supposed to enumerate input-output pairs, and, of course, the complement of a recursively enumerable set may very well fail to be recursively enumerable. So, I agree that it is not very reasonable to complement programs. But, I would welcome some simple-minded examples with relations to see if I can grasp the ideas. Please, show me some interesting properties of relations that can be expressed by inclusions between positive terms.
B – For instance, let M be a a non-empty set. Denote by I_M and E_M the identity and the universal relation on M, respectively. Also, denote by unary $^{-1}$ and binary $|$ the usual operations of reversion and composition of relations. Then, considering a relation r on M, both inclusions $I_M \subseteq r \mid r^{-1}$ and $I_M \subseteq r \mid E_M$ express that r is *total*, i.e., that for all $a \in M$ there exists $b \in M$ satisfying $(a, b) \in r$. Moreover, the

relation $(r \mid r^{-1}) \cap I_M$ determines the subset of I_M consisting of all pairs $(a,a) \in E_M$ such that there exists some $b \in M$ with $(a,b) \in r$. Of course, we have:

$$(r \mid r^{-1}) \cap I_M = (r \mid E_M) \cap I_M. \tag{1.1}$$

This may be expressed by the two obvious inclusions:

$$(r \mid r^{-1}) \cap I_M \subseteq (r \mid E_M) \cap I_M, \tag{1.2}$$

$$(r \mid E_M) \cap I_M \subseteq (r \mid r^{-1}) \cap I_M. \tag{1.3}$$

A – Oh, I see. But, can you be a little bit more precise?
B – Yes, let us define this as a proper formal system. To define the syntax of the *positive relational calculus*, +RC, you start with a countable set Rvar of *relational variables* whose elements are typically denoted by $r, s, t, \ldots$ From Rvar you generate the *positive relational terms*, typically denoted by $R, S, T \ldots$, by using the following grammar:

$$R ::= \mathsf{E} \mid \mathsf{I} \mid r \mid R^{\top} \mid R \sqcap R \mid R \sqcup R \mid R \circ R.$$

For instance, $r \circ r^{\top}$ and $r \circ \mathsf{E}$ are relational terms. An *inclusion* is a formula of the form $R \sqsubseteq S$, where R and S are relational terms. Hence, inclusions (1.2) and (1.3) may be expressed in +RC as:

$$(r \circ r^{\top}) \sqcap \mathsf{I} \sqsubseteq (r \circ \mathsf{E}) \sqcap \mathsf{I}, \tag{1.4}$$

$$(r \circ \mathsf{E}) \sqcap \mathsf{I} \sqsubseteq (r \circ r^{\top}) \sqcap \mathsf{I}. \tag{1.5}$$

A – I'm OK about the syntax. So, let's move on to semantics.
B – Well, terms and inclusions will be interpreted on models. A *model* is simply a system $\mathfrak{M} = \langle M, (r^{\mathfrak{M}})_{r \in \mathsf{Rvar}} \rangle$, with $r^{\mathfrak{M}} \subseteq E_M$, for each relational variable r. The *meaning* of a term R is a binary relation $[\![R]\!]_{\mathfrak{M}}$ on M, defined as usual, by interpreting $\mathsf{E}, \mathsf{I}, {}^{\top}, \sqcap, \sqcup, \circ$ as, respectively, the universal relation E_M,
the identity relation I_M,
reversion, intersection, union and composition of relations. Since an inclusion consists just of a pair of terms, we say that an inclusion *holds* in a model $\mathfrak{M}$, denoted by $\mathfrak{M} \models R \sqsubseteq S$, iff $[\![R]\!]_{\mathfrak{M}} \subseteq [\![S]\!]_{\mathfrak{M}}$. Now, an inclusion is *valid*, denoted by $\models R \sqsubseteq S$, iff it holds in every model $\mathfrak{M}$. Our goal is to decide the valid inclusions of +RC.
A – This does sound interesting. How did you approach this problem?

2 Equational system

Buridanus replies to Asinus by showing him his first approach to validity in +RC*: an equational calculus.*

B – I thought it would be natural to consider an equational calculus.
A – Since you certainly have $\models R \sqsubseteq S$ iff $\models R \sqcap S = R$ iff $\models R \sqcup S = S$, it does look natural, at first sight. But, does it work?
B – I assigned some exercises to my students and left them using inclusions and equalities at will to select an initial set of axioms. Some examples were rather simple.
For instance, they proved inclusions (1.4) and (1.5) as follows:

$$\begin{aligned}(r \circ \mathsf{E}) \sqcap \mathsf{I} &= (r \circ (\mathsf{E} \sqcup r^{\mathsf{T}})) \sqcap \mathsf{I} \\ &= ((r \circ \mathsf{E}) \sqcup (r \circ r^{\mathsf{T}})) \sqcap \mathsf{I} \\ &= ((r \circ \mathsf{E}) \sqcap \mathsf{I}) \sqcup ((r \circ r^{\mathsf{T}}) \sqcap \mathsf{I}).\end{aligned}$$

So, we have $(r \circ r^{\mathsf{T}}) \sqcap \mathsf{I} \sqsubseteq (r \circ \mathsf{E}) \sqcap \mathsf{I}$. Also, they provided:

$$\begin{aligned}(r \circ r^{\mathsf{T}}) \sqcap \mathsf{I} &= (r \circ (r^{\mathsf{T}} \sqcap \mathsf{E})) \sqcap \mathsf{I} \\ &\sqsubseteq ((r \circ r^{\mathsf{T}}) \sqcap (r \circ \mathsf{E})) \sqcap \mathsf{I} \\ &\sqsubseteq (r \circ \mathsf{E}) \sqcap \mathsf{I}.\end{aligned}$$

So, we have $(r \circ \mathsf{E}) \sqcap \mathsf{I} \sqsubseteq (r \circ r^{\mathsf{T}}) \sqcap \mathsf{I}$. Examining these proofs, we were led to axioms like the following ones, valid for any terms R, S, T:

$$\begin{aligned}\mathsf{E} \sqcup R &= \mathsf{E}, \\ R \sqcap \mathsf{E} &= R, \\ R \circ (S \sqcup T) &= (R \circ S) \sqcup (R \circ T), \\ R \circ (S \sqcap T) &\sqsubseteq (R \circ S) \sqcap (R \circ T).\end{aligned}$$

A – Yes, I'll bet even a trivium student could handle such matters.
B – I couldn't agree more. There are, however, some much more complicated examples. For instance, consider the following inclusion $R \sqsubseteq S$, based on Lyndon [3, 4], where a, b, c, d, e, f, g are relational variables:

$$R = a \sqcap (((b \circ c) \sqcap d) \circ (e \sqcap (f \circ g)))$$

and

$$S = b \circ \left(\begin{array}{l} (((b^{\mathsf{T}} \circ a) \sqcap (c \circ e)) \circ g^{\mathsf{T}}) \\ \sqcap \\ (c \circ f) \\ \sqcap \\ (b^{\mathsf{T}} \circ ((a \circ g^{\mathsf{T}}) \sqcap (d \circ f))) \end{array} \right) \circ g.$$

This inclusion looks valid, but not a soul in my class succeeded in proving it by equational means from the initial axioms we selected, neither could they imagine some simple axioms to add. So, they asked me to do it. I am ashamed to confess that neither did I manage to do it. Luckily, the bell rang: I was literally saved by the bell!
A – So far, we have given very few details about your equational calculus. Nevertheless, I might be able to explain why you were having trouble in establishing this inclusion equationally from a nice set of axioms.
B – I can anticipate what you are going to tell me: it is known that the *full* relational calculus is not finitely axiomatizable [7]. Be kind enough, however, to bear in mind that we are dealing with $+\mathsf{RC}$, its positive fragment.
A – If memory serves me right, there is a result of B. Schein [9] that can be used to prove that $+\mathsf{RC}$ is axiomatizable.
B – So, I was on the right track, was I not?
A – Wait, wait. Don't let your enthusiasm carry you away. I'm afraid I've got some bad news for you... To begin with, at a first sight, Schein's result do not guarantee equational axiomatizations, just axiomatizations by universal sentences.

B – Oh dear!
A – Things are worse than that, I should say. There is a result by H. Andréka [1] that states that any relational calculus containing composition and union among its relational operators is not finitely axiomatizable at all. So, you see, if it exists, your beloved equational axiomatization for $+\mathsf{RC}$ will have to be infinite.
B – I guess I see your point. Such an equational axiomatization would not only be infinite, but also rather difficult to describe, let alone to use.

3 Natural deduction system

Desperate, Buridanus puts his conservative mind to work and suggests a second approach to validity in $+\mathsf{RC}$*: natural deduction.*

B – On the other hand, we could use *points* and define a natural deduction system based on them.
A – That might work. I think I have heard about a natural deduction system [14] and a sequent calculus [5] for similar purposes. But, how would you formulate such a system?

3.1 Natural deduction calculus

Buridanus now presents the syntax, semantics and derivation rules of his natural deduction system.

B – To define the syntax of the *natural deduction calculus*, ND, you take a set of *individual variables* Ivar, typically denoted by $x, y, z, \ldots$, besides the positive relational terms. From this we define a new set of expressions, called *formulas*, having the form xRy, where $x, y \in \mathsf{Ivar}$ and R is a term.
A – So, the system will be two-sorted. But you have neither connectives nor quantifiers, just atomic formulas.
B – Yes, but do not forget the operators are hidden inside the terms. So, let us agree to call *simple* those formulas built from atomic terms, i.e., formulas of the form xry.
A – Yes, this terminology sounds good, since the non-simple formulas have occurrences of operators.
B – Formulas will be interpreted on the same class of models as before. Since we now have individual variables, we should assign individuals to them. An *assignment* for the individual variables is a function $g : \mathsf{Ivar} \to M$. The *extension* of a formula xRy on a model $\mathfrak{M}$ is the set of assignments:

$$[\![xRy]\!]_{\mathfrak{M}} := \{g : g \text{ is an assignment and } (gx, gy) \in [\![R]\!]_{\mathfrak{M}}\}.$$

A – As usual, the fact that an assignment $g : \mathsf{Ivar} \to M$ is in $[\![xRy]\!]_{\mathfrak{M}}$ or not hinges only on the values assigned to x and y.
B – In fact, if g and g' agree on x and y, then $g \in [\![xRy]\!]_{\mathfrak{M}}$ iff $g' \in [\![xRy]\!]_{\mathfrak{M}}$. Also, the *extension* of a set Γ of formulas is the set:

$$[\![\Gamma]\!]_{\mathfrak{M}} := \bigcap_{xRy \in \Gamma} [\![xRy]\!]_{\mathfrak{M}}.$$

Say that a formula uSv is a *consequence* of a set of formulas Γ, denoted $\Gamma \models uSv$, iff $[\![\Gamma]\!]_{\mathfrak{M}} \subseteq [\![uSv]\!]_{\mathfrak{M}}$, for every model $\mathfrak{M}$.
A – Now, let me guess. Since we have $\models R \sqsubseteq S$ iff $xRy \models xSy$, we want a set of natural deduction rules such that, for every finite set of formulas $\Gamma \cup \{xSy\}$, we have $\Gamma \models xSy$ iff $\Gamma \vdash_{\mathsf{ND}} xSy$; for then $\models R \sqsubseteq S$ iff $xRy \vdash_{\mathsf{ND}} xSy$.
B – The rules are somewhat as you would expect.

Buridanus draws the introduction rules on the ground using a stick (Table 1).

$$\frac{}{x\mathsf{E}y} \qquad \frac{}{x\,\mathsf{I}\,x}$$

$$\frac{xRy \quad xSy}{x(R \sqcap S)x}$$

$$\frac{xRy}{x(R \sqcup S)y} \qquad \frac{xSy}{x(R \sqcup S)y}$$

$$\frac{xRy}{yR^{\top}x}$$

$$\frac{xRz \quad zSy}{x(R \circ S)y}$$

TABLE 1. ND Introduction Rules.

A – I can see that these introduction rules are quite similar to the ones for classical logic [8, 11]. The only difference is that the connectives and quantifiers are hidden in the relational terms, as you said. Following this line, I would imagine you will need some restriction on the elimination rule for composition.
B – Indeed, the elimination rules are also reminiscent of those for classical logic (cf. Table 2). In particular, the elimination rules for union and for composition discharge hypotheses, and the restrictions on the latter are meant to ensure soundness.
A – I see. The elimination rule for $\sqcup$ captures the idea: if $\Gamma \cup \{xRy\} \vdash_{\mathsf{ND}} uTv$ and $\Gamma \cup \{xSy\} \vdash_{\mathsf{ND}} uTv$, then $\Gamma \cup \{x(R \sqcup S)y\} \vdash_{\mathsf{ND}} uTv$. But, wait; haven't you made a mistake in the elimination rule for I?
B – No, not really. It does look somewhat strange. But, it is sound and I need this form to establish reflexivity, symmetry and transitivity of I.

3.2 Proving Lyndon's inclusion in ND

Buridanus now shows how to use his system to establish Lyndon's inclusion.

A – How could you use your system to solve your problem of establishing Lyndon's inclusion $R \sqsubseteq S$ mentioned earlier (Section 2)? I guess you could try to break down

$$\frac{xRz \quad y \mathsf{I} z}{xRy}$$

$$\frac{x(R \sqcap S)y}{xRy} \qquad \frac{x(R \sqcap S)y}{xSy}$$

$$\frac{\begin{array}{c}[xRy]^i \\ \vdots \\ uTv\end{array} \quad \begin{array}{c}[xSy]^i \\ \vdots \\ uTv\end{array} \quad x(R \sqcup S)y}{uTv}\,i$$

$$\frac{xR^{\intercal}y}{yRx}$$

$$\frac{\begin{array}{c}[xRz]^j \quad [zSy]^j \\ \vdots \\ uTv\end{array} \qquad x(R \circ S)y}{uTv}\,j, \quad z \text{ is new}$$

TABLE 2. ND Elimination Rules.

the left-hand side xRy into simpler subformulas and then reassemble them to obtain the right-hand side xSy. This approach looks pretty straightforward for $\sqcap$; but what about $\sqcup$ and $\circ$, their elimination rules do not seem adequate for this task.

B – A syntax driven strategy for establishing an inclusion $R \sqsubseteq S$ using ND has three steps. First, analyze the structure of formula xRy, corresponding to the left-hand side term, breaking it into simple subformulas by using introduction rules backwards.

A – In this process some new hypotheses may appear. These new hypotheses will have to be derived or discharged afterwards.

B – You are quite right. This will be the third step. The second step is to synthesize formula xSy, corresponding to the right-hand side term: apply introduction rules forwards to the simple subformulas obtained in the first step, to construct xSy. Third, use this raw material to obtain the derivation by applying elimination rules forwards to establish or discharge hypotheses other than xRy.

A – Let's see how to use your strategy to establish Lyndon's inclusion.

B – The left-hand side R has the form $a \sqcap (R_1 \circ R_2)$, where $R_1 = (b \circ c) \sqcap d$ and $R_2 = e \sqcap (f \circ g)$. Thus, you have the analysis of xRy in Figure 1.

A – In this manner, you obtain the syntactic tree of the left-hand side xRy.

B – Yes, and from this you obtain the following set of simple subformulas corresponding to xRy:

$$\Sigma = \{xay, xbv, vcu, xdu, uey, ufw, wgy\}.$$

This is the material one has available to construct the right-hand side xSy. One

$$\{xRy\}$$

$$\{xay, x(R_1 \circ R_2)y\}$$

$$\{xay, x((b \circ c) \sqcap d)u, u(e \sqcap (f \circ g))y\}$$

$$\{xay, x(b \circ c)u, xdu, uey, u(f \circ g)y\}$$

$$\{xay, xbv, vcu, xdu, uey, ufw, wgy\}$$

FIG. 1. Analysis of xRy.

should synthesize xSy using the rules to derive it from the set Σ. Now, S has the form $b \circ (T_1 \sqcap T_2 \sqcap T_3) \circ g$, where $T_1 = ((b^{\mathsf{T}} \circ a) \sqcap (c \circ e)) \circ g^{\mathsf{T}}$, $T_2 = c \circ f$, and $T_3 = b^{\mathsf{T}} \circ ((a \circ g^{\mathsf{T}}) \sqcap (d \circ f))$. Thus, one can derive xSy from Σ in a stepwise manner as follows. Simple formulas xbv and wgy are already in Σ. It remains to derive the subformulas vT_1w, vT_2w, and vT_3w.

A – Just a moment! Why this choice of variables, say v and w? Why not fresh variables: v' and w'?

B – The rule for $\circ$ introduction does not require this. But, you do have a point. One should use only the material available in Σ, so only the variables x, y, v, u, w. Now, this choice appears to be adequate, because we already have xbv and wgy in Σ, and the variable prefixing terms T_1, T_2, and T_3 should match the right one of xbv, while the variable prefixing T_1, T_2, and T_3 should match the left one of wgy. You see, one needs some symbolic foresight.

A – In this case it is OK. But you could have several choices, which would complicate the choice of such matchings.

B – You are quite right, but I am just presenting a strategy, I am not yet concerned with automating it.

A – I see, as long as Your Enlightened Majesty can do it... But OK, you are going to construct xSy bottom-up. I can see how to derive $vT_2w = v(c \circ f)w$ from vcu and ufw by $\circ$-introduction:

$$(\circ\, i)\ \frac{vcu \quad ufw}{v(c \circ f)w}$$

B – The derivations of vT_1w and vT_3w are similar, only slightly longer. One can then combine these derivations into one of $v(T_1 \sqcap T_2 \sqcap T_3)w$ by $\sqcap$-introductions and finally obtain one for $xSy = x(b \circ ((T_1 \sqcap T_2 \sqcap T_3) \circ g))y$ by $\circ$-introductions.

A – So, you obtain a derivation Π of the right-hand side xSy, but from Σ, rather than from the left-hand side xRy.

B – Now comes the final step. One can now transform the derivation Π showing that $\Sigma \vdash_{\mathsf{ND}} xSy$ into a derivation $xRy \vdash_{\mathsf{ND}} xSy$ by applying elimination rules forwards to establish or discharge hypotheses other than xRy. For instance, to derive xay, it suffices to apply $\sqcap$-elimination to xRy:

$$(\sqcap e)\ \frac{x(a \sqcap (R_1 \circ R_2))y}{xay}$$

A – This is straightforward for xay. But what about the other hypotheses, mainly those with new variables, such as ufw?
B – One can discharge these hypotheses by relying on the syntactic structure of xRy. In this particular case, one discharges both ufw and wgy in favor of $u(f \circ g)y$:

$$(\circ e)\ \frac{\begin{array}{c}[ufw]^1\ [wgy]^1\\ \Pi\\ xSy\end{array} \qquad u(f \circ g)y}{xSy}\,1$$

A – I see, you apply the $\circ$-elimination to transform the derivation Π with set of hypotheses $\Sigma' \cup \{ufw, wgy\}$ into another one Π' with set of hypotheses $\Sigma' \cup \{u(f \circ g)y\}$, while keeping the conclusion. And now, you can derive both uey and $u(f \circ g)y$ from $uR_2y = u(e \sqcap (f \circ g))y$:

$$\frac{\begin{array}{c}\Sigma'\\ \Pi'\\ xSy\end{array} \quad (\sqcap e)\ \dfrac{u(e \sqcap (f \circ g))y}{u(f \circ g)y}}{xSy}$$

B – Indeed, similarly one can discharge both xbv and vcu in favor of $u(b \circ c)y$ and then derive both $u(b \circ c)y$ and xdu from xR_1u. Next, one can discharge both xR_1u and uR_2y in favor of $u(R_1 \circ R_2)y$, which is obtained from xRy by $\sqcap$-elimination.
A – I see, you climb the analysis tree of the left-hand side xRy, using the elimination rule where you had previously applied backwards the corresponding introduction rule. In this manner, you manage to transform the derivation of xSy from Σ into a derivation establishing that $xRy \vdash_{\mathsf{ND}} xSy$.

3.3 Soundness and completeness

Now, Buridanus is happy because he is able to provide a complete proof for his favorite problem. But Asinus still has some doubts.

A – OK, you can use your calculus to establish Lyndon's inclusion. But, I notice that the symbols E, I and $\sqcup$ do not appear in the terms of this inclusion. Can you adapt your strategy to cope with them? For instance, if you want to apply the above strategy to prove an inclusion as:

$$r \circ (s \sqcup t) \sqsubseteq (r \circ s) \sqcup (r \circ t),$$

you have to consider trees containing more than one leaf (Figure 2).
B – Indeed, I should adapt this strategy to cover such cases. It will become more involved, but I think it can be done. In fact, I can establish soundness and completeness of this natural deduction calculus.

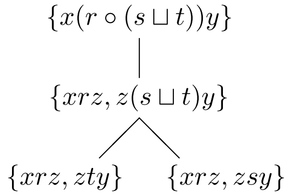


FIG. 2. Analysis of $x(r \circ (s \sqcup t))y$.

PROPOSITION 3.1 (Soundness of ND)
If $\Gamma \vdash_{\mathsf{ND}} uSv$, then $\Gamma \models uSv$.

PROOF. Given a model $\mathfrak{M}$, we check that $[\![\Gamma]\!]_{\mathfrak{M}} \subseteq [\![uSv]\!]_{\mathfrak{M}}$, whenever $\Gamma \vdash_{\mathsf{ND}} uSv$, by induction on the construction of the derivation. ∎

B – We can leave the details of the inductive proof to the students. In doing the case for $\circ$-elimination, they might see the reason for the restriction on this rule.
A – Well, as usual, soundness looks pretty straightforward. But, what about completeness?
B – I think one can establish completeness by relying on the subformula property of normal derivations.
A – I presume that all derivations in ND can be normalized.

LEMMA 3.2 (Normalization in ND)
Each derivation in ND can be reduced to a normal one.

PROOF. For each formula that is the conclusion of an introduction rule and major premise of an elimination rule, we have an appropriate reduction, as usual. ∎

B – Now I can prove completeness, showing that the strategy I outlined does work in general, as you asked for.

THEOREM 3.3 (Completeness of ND)
Let Γ be a finite set of formulas. If $\Gamma \models uSv$, then $\Gamma \vdash_{\mathsf{ND}} uSv$.

PROOF. The proof relies on two constructions and their properties:

1. Analysis tree:
   (a) Construct a tree $\mathcal{T}_\Gamma$ analyzing the set Γ of hypotheses into simple subformulas and formulas of the form $u \,|\, v$, by using introduction rules backwards.
   (b) Show that $\Gamma \vdash_{\mathsf{ND}} uSv$ iff $\Sigma \vdash_{\mathsf{ND}} uSv$, for each leaf Σ of this tree $\mathcal{T}_\Gamma$.
2. Canonical model and assignment:
   (a) Define a canonical model $\mathfrak{C}_\Sigma$ and assignment g_Σ, for each leaf Σ of the tree $\mathcal{T}_\Gamma$.
   (b) Show that, for every formula uRv, $g_\Sigma \in [\![uRv]\!]_{\mathfrak{C}_\Sigma}$ iff $\Sigma \vdash_{\mathsf{ND}} uRv$.

One can now use these ideas to prove completeness of ND. Assume $\Gamma \nvdash_{\mathsf{ND}} uSv$. To show $\Gamma \not\models uSv$, it suffices to provide a model $\mathfrak{M}$ such that $[\![\Gamma]\!]_{\mathfrak{M}} \not\subseteq [\![uSv]\!]_{\mathfrak{M}}$. Consider the analysis tree $\mathcal{T}_\Gamma$ of Γ constructed in 1(a) above. For some leaf Σ of $\mathcal{T}_\Gamma$, by the property in 1(b) above, we have:

$$\Sigma \nvdash_{\mathsf{ND}} uSv; \text{ but } \Sigma \vdash_{\mathsf{ND}} xRy, \text{ for every } xRy \in \Gamma.$$

Consider the canonical model $\mathfrak{C}_\Sigma$ and assignment g_Σ constructed in 2(a) above. Then, by the property in 2(b) above, we have:

$$g_\Sigma \in [\![xRy]\!]_{\mathfrak{C}_\Sigma}, \text{ for every } xRy \in \Gamma; \text{ but } g_\Sigma \notin [\![uSv]\!]_{\mathfrak{C}_\Sigma}.$$

Hence $g_\Sigma \in [\![\Gamma]\!]_{\mathfrak{C}_\Sigma}$, but $g_\Sigma \notin [\![uSv]\!]_{\mathfrak{C}_\Sigma}$; therefore $\Gamma \not\models uSv$. ∎

A – Let us return to items 1 and 2 of the Completeness result. The construction of the analysis tree of a set of formulas is similar to the idea you used in your strategy before (Section 3.2). You construct a tree analyzing each formula of a given set into its simple parts.

B – Yes, you start with a tree consisting only of a given non-empty set Γ of formulas as its root.

A – Then you will use an introduction rule for the main symbol of a formula to break it down into subformulas.

B – Yes, at each stage you take a node Δ that is not yet a leaf, select a non-basic formula xRy in it and add sons where you replace formula xRy in Δ by the premises of the introduction rule for the main symbol of xRy.

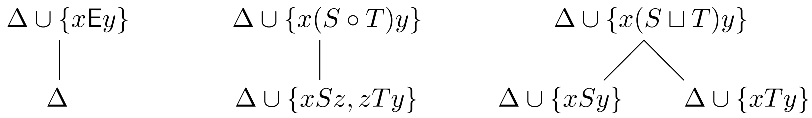
A – So, such a node will have only one son, except on the case of $\sqcup$, when it will have two sons.

B – Indeed, the *expansion* of a node $\Delta \cup \{xRy\}$ in a tree will be as you may expect at this point. Some typical cases are as follows.

(E) If xRy is $x\mathsf{E}y$, then it has a single son: Δ ($x\mathsf{E}y$ is erased).

($\circ$) If xRy is $x(S \circ T)y$, then it has a single son: $\Delta \cup \{xSz, zTy\}$, where z is the first new variable (distinct from those used so far).

($\sqcup$) If xRy is $x(S \sqcup T)y$, then it has two sons: $\Delta \cup \{xSy\}$ and $\Delta \cup \{xTy\}$.



In this way, you can construct a tree analyzing each set into its simple parts. Also, this expansion procedure provides all the conditions for derivability we need.

Lemma 3.4 (Expansion)
Given a node Δ in an analysis tree. Then, for every formula uPv, we have $\Delta \vdash_{\mathsf{ND}} uPv$ iff $\Delta' \vdash_{\mathsf{ND}} uPv$, for every son Δ' of Δ.

Proof. By construction, using the rules of introduction ($\Rightarrow$) and elimination ($\Leftarrow$). ∎

A – Now, you can use induction to extend the Lemma 3.4 to analysis trees.

Corollary 3.5 (Analysis tree)
Given a finite set Γ of formulas, consider an analysis tree $\mathcal{T}_\Gamma$ of Γ. Then, for every formula uSv, we have $\Gamma \vdash_{\mathsf{ND}} uSv$ iff $\Sigma \vdash_{\mathsf{ND}} uSv$, for every leaf Σ of $\mathcal{T}_\Gamma$.

B – For instance, consider the left-hand side xRy of the above Lyndon's inclusion (Section 2). We have already constructed its analysis tree (Figure 1). It consists of a single leaf $\Sigma = \{xay, xbv, vcu, xdu, uey, ufw, wgy\}$. For such terms without union, the analysis tree will become a linear unary tree.

A – This is the raw material you use for the canonical construction.
B – Quite right. As usual, the points of the canonical model will be equivalence classes of individual variables.
A – I see. Given a nonempty basic set Σ of formulas, you first define the canonical relation $\sim_\Sigma$ on Ivar by $x \sim_\Sigma y$ iff $\Sigma \vdash_{\mathsf{ND}} x\,\mathsf{I}\,y$.
B – As you may guess, this canonical relation is an equivalence and the *canonical model* $\mathfrak{C}_\Sigma := \langle C_\Sigma, (r_\Sigma)_{r\in\mathsf{Rvar}}\rangle$ is defined by $C_\Sigma := \mathsf{Ivar}|_{\sim\Sigma}$ and $r_\Sigma := \{(\widetilde{x}^\Sigma, \widetilde{y}^\Sigma) : xry \in \Sigma\}$. Also, the *canonical assignment* $g_\Sigma : \mathsf{Ivar} \to C_\Sigma$ is defined by $g_\Sigma x := \widetilde{x}^\Sigma$. Finally, you can see the basic property of the canonical construction, relating satisfaction to derivability.

Proposition 3.6 (Canonical construction for ND)
Given a set Σ of basic formulas, consider its canonical model $\mathfrak{C}_\Sigma$ and assignment $g_\Sigma : \mathsf{Ivar} \to M_\Sigma$. Then, for every term uRv, $g_\Sigma \in [\![uRv]\!]_{\mathfrak{C}_\Sigma}$ iff $\Sigma \vdash_{\mathsf{ND}} uRv$.

Proof. The case $u\mathsf{E}v$ is clear, the other cases are handled by induction, using the Normalization Lemma 3.2. ($\Rightarrow$) By induction on a normal derivation, which starts with introduction rules. ($\Leftarrow$) By induction on the structure of term uRv. ∎

A – OK, I'll grant you that the plan above to prove completeness will work. But, it looks like this approach is unnecessarily complex, isn't it?
B – What do you mean? Pray, enlighten me.
A – You see, natural deduction is somewhat complex with all this business of discharging assumptions. Moreover, the derivations are shaped like trees.
B – And how else should they be shaped? Like circles, perhaps?
A – Oh, come on, cut the sarcasm, I mean it. The problem at hand concerns valid inclusions between terms. For this specific purpose, I can envisage something simpler, taking the essentials of your ideas: linear derivations in a graph calculus.
B – ?!?!

4 Graph system

Asinus presents his alternative approach to validity in +RC*: reasoning with graphs.*

A – As long as you are using points, we might as well view xRy as a graph.
B – I am afraid I do not quite see what you are hinting at.
A – The basic idea is to use (directed arc-labeled pseudo multi) graphs having two distinguished nodes and arcs labeled by positive relational terms to represent a binary relation. The label of an arc represents a restriction associated to the relation defined by the label. A path from a node to another represents a restriction associated to the composition of the corresponding relations. Two paths with the same start and end points and sharing no nodes represent a restriction associated to the intersection of the corresponding relations.
B – I am beginning to see it: parallel arcs represent intersection and consecutive ones represent composition. With the two distinguished nodes you get an input-output relation. But, wait a moment, this would be semantics! What we want is a proof system.

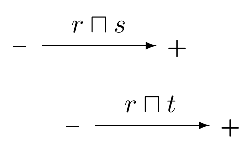
A – I'm coming to that: how one can use graphs to prove inclusions formally. I guess we can base the construction of a proof system on a folklore theorem about completeness via normal forms. Roughly speaking, the theorem states that any deductive system that can be used both to derive normal forms of formulas and to prove that a formula is a logical consequence of another, both in normal form, is simply complete.
B – Asinus, what is a simply complete system?
A – A deductive system is *simply complete* when one can derive a formula ψ from a formula θ whenever ψ is a logical consequence of θ. Since we are aiming at deriving xSy from xRy whenever $xRy \models xSy$, this is enough for our purposes. So, we just need rules to transform a graph into a graph in normal form and to derive a graph in normal form from another one, also in normal form.
B – A good notion of normal form for graphs should be graphs whose arcs are labeled with atomic terms. I can easily see how to adapt the introduction and elimination rules of ND to do this job.
A – Let us put the graphs G_R and G_S associated to the terms R and S of Lyndon's inclusion in this normal form.

> *Starting from the graph G_R associated to term R of Lyndon's inclusion, Asinus draw a sequence of graphs, showing the transformation of G_R into a graph in normal form (Figure 3). Analogously for S (Figure 4).*

B – That is the easy part. I ask you: how to compare graphs in normal form, to derive one from the other?
A – Since graphs are just relational structures, it is natural to use homomorphisms between graphs to compare them. In the case of Lyndon's inclusion, we have to find a homomorphism from the normal form of G_S into the normal form of G_R, since a graph with fewer restrictions will define a bigger relation. But this is an easy task, because the normal form of G_R has distinct labels in distinct arcs.
B – By George, this is quite impressive. With this kind of sorcery, even trivium students could do it. But, Asinus, I have one further question before we proceed. I do not quite see how you would represent union in graphs.
A – This is a fine point I glossed over. A graph is actually a set of slices, corresponding to alternatives. Until now we just have examples of graphs with a single slice. In the general case, one can represent union by considering that each graph represents a restriction associated to the union of the relations corresponding to its slices. This will also have some consequences in defining a proper notion of homomorphism between graphs in general.
B – May we examine the inclusion

$$(r \sqcap s) \sqcup (r \sqcap t) \sqsubseteq r \sqcup q$$

as an example?
A – The rule of $\sqcup$-elimination applied to the graph G associated to the left-hand side of this inclusion will lead to a graph with two slices:



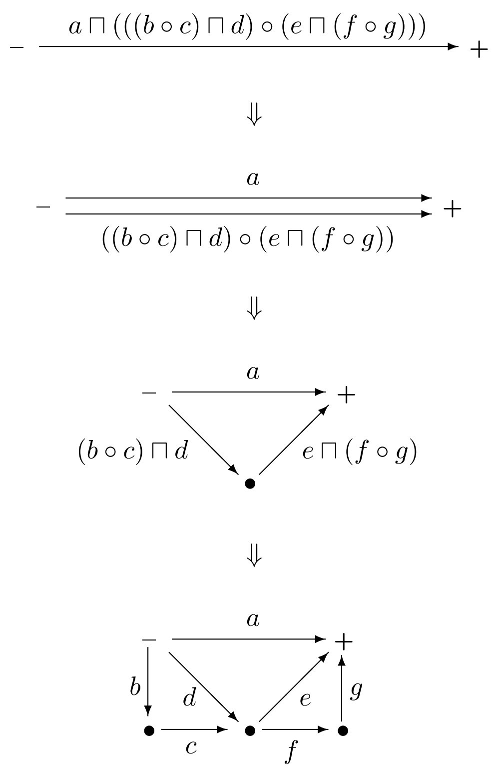
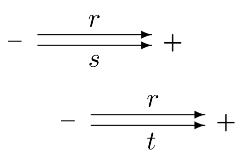


FIG. 3. Transforming G_R into a graph in normal form.

We can keep your notion of normal form and obtain the following graph with two slices as the normal form of G:



B – But what about the concept of homomorphism for these graphs with more than one slice?
A – Homomorphism between graphs with slices should subsume homomorphisms between slices we were using until now. The normal form of the graph H associated to the right-hand side of this inclusion has also two slices:

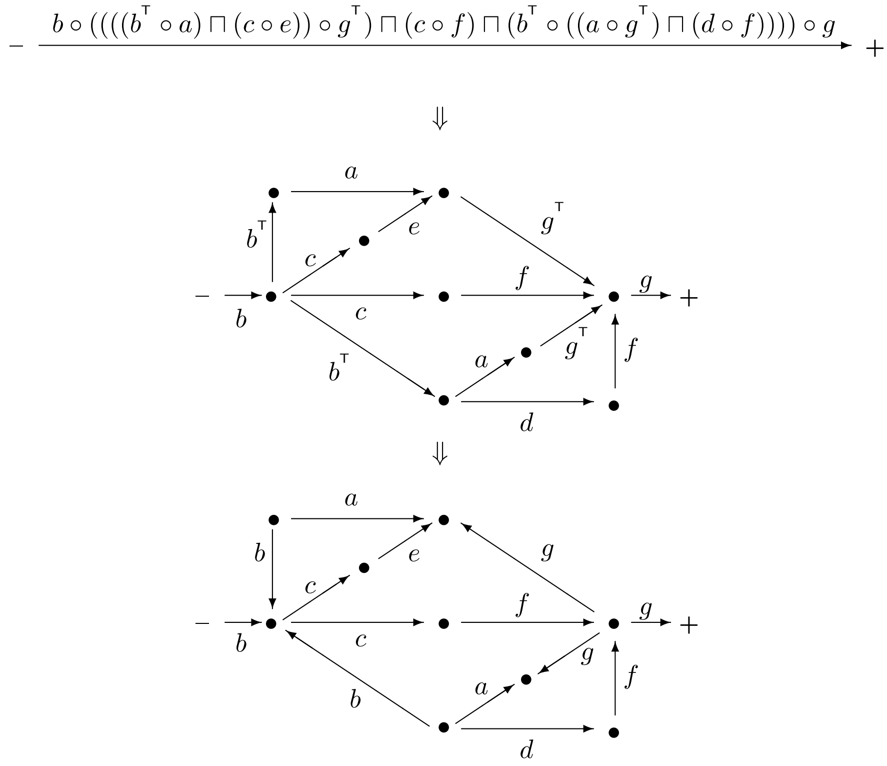
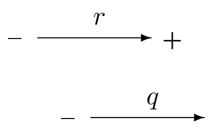


FIG. 4. Transforming G_S into a graph in normal form.



We can see that the inclusion is valid observing that, for each slice S in the normal form of G, there is a slice S' in the normal form of H and a slice homomorphism from S' into S.
B – Nice! But how can these ideas be formalized as a proper logical system, something respectful that could be taught at a decent university?

4.1 Positive graph calculus: syntax and semantics

Asinus formalizes his proposal by presenting what he calls the positive graph calculus.

A – To define the syntax of the positive graph calculus, $+\mathsf{RG}$, you take a countable set of nodes, typically denoted $u, v, w, \ldots$, besides the positive relational terms. A *slice*

is a quadruple $S = (N, A, x, y)$, where $N \neq \emptyset$ is a set of nodes, $A \subseteq N \times \mathsf{Labels} \times N$ is a set of arcs, with Labels being the set of positive relational terms, and $x, y \in N$. A *graph* $G = (S_i)_{i \in I}$ is a finite set of slices. As you pointed out, Buridanus, the semantics of $+\mathsf{RG}$ is directly adapted from that of ND. A model is again of the form $\mathfrak{M} = \langle M, (r^{\mathfrak{M}})_{r \in \mathsf{Rvar}} \rangle$, and assigns a binary relation $[\![R]\!]_{\mathfrak{M}}$ on M to each relational term R. Given a slice $S = (N, A, x, y)$, an *assignment for* S in $\mathfrak{M}$ is $g : N \to M$ such that $(gu, gv) \in [\![R]\!]_{\mathfrak{M}}$, for each $uRv \in A$. The *meaning of slice* S in $\mathfrak{M}$ is $[\![S]\!]_{\mathfrak{M}} := \{(gx, gy) \in M \times M : g \text{ is an assignment for } S \text{ in } \mathfrak{M}\}$. The *meaning of graph* $G = (S_i)_{i \in I}$ in $\mathfrak{M}$ is $[\![G]\!]_{\mathfrak{M}} = \bigcup_{i \in I} [\![S_i]\!]_{\mathfrak{M}}$.
B – Very good! You have formalized the use of graphs to establish an inclusion $R \sqsubseteq S$ between positive relational terms.
A – Also, as we have done in our Lyndon example, we associate to each term R its graph $G_R = (\{x, y\}, \{xRy\}, x, y)$. Then, we will have, for each model $\mathfrak{M}$, $\models R \sqsubseteq S$ iff $[\![G_R]\!]_{\mathfrak{M}} \subseteq [\![G_S]\!]_{\mathfrak{M}}$.

4.2 Positive graph calculus: basic rules

B – I think I can imagine some rules to transform graphs. For instance, I would expect a rule reversing an arc $uR^{\intercal}v$ to vRu. Also, if I understood correctly the idea of slices, you would have a rule replacing an arc $uR \sqcup Sv$ by two slices: one with uRv and another one with uSv. But, how do you formulate them in general?
A – Indeed, the introduction and elimination rules are quite what one would expect. Before you ask me again, here are the rules.

Asinus states the rules of the calculus (Tables 3 and 4).

A – Rule Conv states that the meaning of a graph does not change by replacing an arc $uR^{\intercal}v$ by vRu, inside a slice where it occurs, leaving the rest of the graph untouched. Rule Int states that the meaning of a graph does not change by replacing an arc $uR \sqcap Sv$ by two others, uRv and uSv, inside a slice where it occurs, leaving the rest of the graph untouched. Rule Comp states that the meaning of a graph does not change by replacing an arc $uR \circ Sv$ by two others, uRw and wSv, with a new node w, inside a slice where it occurs, leaving the rest of the graph untouched. Rule Uni states that the meaning of a graph does not change by replacing a slice S_1 having occurrence of an arc $uR \sqcup Sv$, by two other slices S_2 and S_3, each one of them obtained from S_1 by replacing the arc $uR \sqcup Sv$ by a new arc: uRv for S_2 and uSv for S_3, leaving the rest of the graph untouched.
B – Now, it is clear what you said earlier about slices and union. There is however something still bothering me concerning slices and connectedness. Must every node be reachable from one of the distinguished nodes?
A – In many cases, a slice is indeed connected in this sense, but this is not required. In fact, I was about to use the name 'component' for a slice. The change was exactly to avoid such misunderstandings. Observe rule Univ. It states that the meaning of a graph does not change by erasing an arc labeled by E from a slice, leaving the rest of the graph untouched.
B – Yes, an application of this rule may disconnect the slice.
A – Finally, rule Iden states that the meaning of a graph does not change by erasing an arc $u\,\mathsf{I}\,v$ and node u, and renaming the slice where they occur, redirecting arcs ac-

Conv	$\dfrac{N, A + uR^{\top}v, x, y}{N, A + vRu, x, y}$
Int	$\dfrac{N, A + uR \sqcap Sv, x, y}{N, A + uRv + uSv, x, y}$
Comp	$\dfrac{N, A + uR \circ Sv, x, y}{N + w, A + uRw + wSv, x, y}$, if $w \notin N$
Uni	$\dfrac{N, A + uR \sqcup Sv, x, y}{(N, A + uRv, x, y) \;\; (N, A + uSv, x, y)}$
Univ	$\dfrac{N, A + u\mathsf{E}v, x, y}{N, A, x, y}$
Iden	$\dfrac{N, A + u\,\mathsf{I}\,v, x, y}{\mathtt{ren}_u^v N, \mathtt{ren}_u^v A, \mathtt{ren}_u^v x, \mathtt{ren}_u^v y}$

TABLE 3. Introduction/Elimination rules.

cordingly. This rule uses the function $\mathtt{ren}_u^v$, rename u to v, described by the following definitions.

$$\mathtt{ren}_u^v w = \begin{cases} v & \text{if } w = u, \\ w & \text{otherwise.} \end{cases}$$

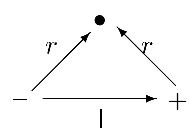
Given arbitrary set of nodes and arcs N and A, respectively, set $\mathtt{ren}_u^v N = \{\mathtt{ren}_u^v w : w \in N\}$ and $\mathtt{ren}_u^v A = \{\mathtt{ren}_u^v wR\mathtt{ren}_u^v w' : wRw' \in A\}$.

B – I think I can use an example to clarify the application of this rule.

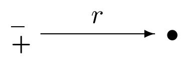
A – Let me see... We can use the inclusion (1.4) you proposed to your students:

$$(r \circ r^{\top}) \sqcap \mathsf{I} \sqsubseteq (r \circ \mathsf{E}) \sqcap \mathsf{I}.$$

B – Applying rules Int, Comp and Conv to the graph associated to the left-hand side, we obtain a graph with an arc $x\,\mathsf{I}\,y$:



A – An application of rule Iden to this graph will identify nodes x and y, distinguished as − and +, respectively, in the figure, leading to a graph having the same node as input and output:



B – This is a bit odd, but the result is a graph in normal form, according to my definition.
A – The derivation of the normal form of the graph associated to the righthand side will lead to the same graph. Thus, we proved, in fact, the equality (1.1), which started our discussion, using the graph calculus.
B – Ah, these transformation rules preserve the meaning of a graph, so they can be applied in both directions and if we define, as I suggested, a graph G is in *normal form*, NF, if the label of every arc in G is atomic, it is clear, by induction, that every graph can be transformed to an equivalent one in NF by application of the elimination rules.

LEMMA 4.1 (Normal form)
Every graph G can be transformed to a graph NFG in NF by application of the elimination rules; moreover, graphs G and NFG have the same meaning in each model.

A – Our capital rule, GrHom, states that a graph homomorphic to a graph G can be inferred from G (Table 4). This idea of comparison by homomorphism can be naturally introduced in stages, for slices and for graphs. Given slices $S = (N, A, x, y)$ and $S' = (N', A', x', y')$, a *slice homomorphism* $\phi : S \to S'$ is a function $\phi : N \to N'$ that preserves distinguished nodes and arc labels, i.e., $\phi x = x'$, $\phi y = y'$, and for each arc $uRv \in A$, we have $\phi u R \phi v \in A'$. Given graphs G and H, let's say that G is *homomorphic to* H, noted $G \leftarrow H$, if for each slice S of G, there exist a slice S' of H and a slice homomorphism $\phi : S' \to S$.

$$\mathsf{GrHom} \quad \frac{G}{H}, \text{ if } G \leftarrow H$$

TABLE 4. Graph homomorphism rule.

Of course, this graph homomorphism rule can be applied only in one direction. Indeed, consider graphs $G = (\{x\}, xrx, x, x)$ and $H = (\{x\}, \emptyset, x, x)$, and the identity function ϕ on $\{x\}$. Obviously, $\phi : H \to G$ is a slice homomorphism and, hence, $G \leftarrow H$. But given any model $\mathfrak{M}$ such that $M = \{a, b\}$ and $r^{\mathfrak{M}} = \{(a, a)\}$, we have $[\![G]\!]_{\mathfrak{M}} = \{(a, a)\}$ and $[\![H]\!]_{\mathfrak{M}} = M \times M$, so $[\![H]\!]_{\mathfrak{M}} \not\subseteq [\![G]\!]_{\mathfrak{M}}$.

4.3 Positive graph calculus: structural rules

> *Now Asinus is happy because he is able to adapt Buridanus strategy to prove inclusions in* ND *into a simple and playful formal system. Buridanus, however, is still not quite happy.*

B – Now, coming to think of it, I guess I will have to take back what I said about even trivium students being able to use this machinery. The idea of graph homomorphism and the corresponding rule look like a real Pons Asinorum, if you will pardon me the pun.
A – We need to apply the graph homomorphism rule at most once. Another advantage of this rule is that we have a normal form for proofs. I do however grant your point about its complexity. You'll be happy to know that there are alternative simpler ways to present these matters.
B – Do you mean you can simulate this rule by other rules that are easier to grasp and to apply?
A – Indeed, a simpler version of the graph homomorphism rule is its version SlHom for slices (Table 5), namely you can replace slice S by T if you have a slice homomorphism $\phi : T \to S$.

SlHom	$\frac{S}{T}$, if $\phi : T \to S$

TABLE 5. Slice homomorphism rule.

A – Another simple one-directional rule AddSl states that you can always add slices (Table 6).

AddSl	$\frac{S}{S \;\; S'}$

TABLE 6. Slice addition rule.

LEMMA 4.2 (Reduction of graph homomorphism)
Rule GrHom can be simulated by the rules SlHom and AddSl.

B – I see that the slice rules are special cases of the graph homomorphism rule. So, you explain the concept of homomorphic graphs by breaking it down into two simpler rules.
A – Indeed, and I can do even better. I can explain slice homomorphisms in terms of some simple manipulations on slices. Consider the following rules about splitting a

node, erasing an isolated non-distinguished node, and erasing an arc (Table 7). Rule SpltNd states that the addition of a new node v having adjacent to it the same arcs as a node u, inside a slice where it occurs, leaving the rest of the graph untouched, does not alter the meaning of the graph. This rule uses the function spl_v^u, split u with v, transforming sets of arcs and nodes, defined by:

$$\begin{aligned}\mathsf{spl}_v^u A \quad = \quad & A \cup \{vRw : uRw \in A\} \cup \{wRv : wRu \in A\} \cup \\ & \{vRv : uRu \in A\}\end{aligned}$$

Rule EraNd states that the meaning of a graph does not change by erasing an isolated node that is not distinguished in a slice. A node is *isolated* when it is not linked to another node by an arc. Rule EraArc states that the meaning of a graph obtained by erasing an arc in a slice where it occurs, leaving the rest of the graph untouched, contains the meaning of the original one.

SpltNd	$\dfrac{N, A, \mathtt{ren}_v^u x, \mathtt{ren}_v^u y}{N + v, \mathsf{spl}_v^u A, x, y}$, if $v \notin N$
EraNd	$\dfrac{N, A, x, y}{N - u, A, x, y}$, if u is isolated and $u \notin \{x, y\}$
EraArc	$\dfrac{N, A, x, y}{N, A - uRv, x, y}$

TABLE 7. Slice structural rules.

B – Again, I see that the slice manipulation rules SpltNd, EraNd and EraArc are special cases of the slice homomorphism rule. So, you were able to explain slice homomorphism by breaking it down into more basic rules.
A – Indeed, when you have a slice homomorphism $\phi : T \to S$, you can construct S from T by splitting nodes and erasing arcs and isolated non-distinguished nodes.
B – Let us see an example. Considering the slices $T = (\{a, b, c\}, \{arc, brc\}, a, b)$ and $S = (\{c, d\}, \{drc\}, d, d)$, the function ϕ given by $\phi(a) = \phi(b) = d$ and $\phi(c) = c$ is a slice homomorphism $\phi : T \to S$. How can you construct T from S?
A – One can obtain T from S by first successively splitting node d into two new nodes a and b, then erasing arc drc and, finally, erasing the isolated non-distinguished node d, as follows.

$$
\begin{array}{cl}
S = (\{c,d\},\{drc\},d,d) & \\
\downarrow \mathsf{SpltNd} & (d = \mathtt{ren}^d_a a,\ d = \mathtt{ren}^d_a d) \\
(\{a,c,d\},\{drc,arc\},a,d) & \\
\downarrow \mathsf{SpltNd} & (a = \mathtt{ren}^d_b a,\ d = \mathtt{ren}^d_b b) \\
(\{a,b,c,d\},\{drc,arc,brc\},a,b) & \\
\downarrow \mathsf{EraArc} & (\text{erase arc } drc) \\
(\{a,b,c,d\},\{arc,brc\},a,b) & \\
\downarrow \mathsf{EraNd} & (\text{erase isolated node } d) \\
T = (\{a,b,c\},\{arc,brc\},a,b) &
\end{array}
$$

Proposition 4.3 (Reduction of slice homomorphism)
Rule SlHom can be simulated by the rules SpltNd, EraNd and EraArc.

B – So, you have a more palatable version for the infamous graph homomorphism rule GrHom.

Corollary 4.4 (Simulation of graph homomorphism)
Rule GrHom can be simulated by the rules SlHom, AddSl, SpltNd, EraNd and EraArc.

A – Finally, to be completely formal, we still need to define explicitly a notion of derivation. This is the usual one. We say that a graph H is *derived* from a graph G, noted $G \vdash H$, when there is a sequence of graphs $(G_0, \ldots, G_n)$, such that $G_0 = G$, $G_n = H$, and each G_i, $i = 1, \ldots, n$, is a consequence of some G_j, with $j < i$, by application of some rule.
B – Since every rule in the system has just one premise, you have linear derivations, as you had promised.

> *Some time later, Buridanus meets Asinus and tells him about the enormous success that the positive graph calculus is having among his students and how they are delighted to show $G_R \vdash G_S$ as a playful exercise. But he still has some questions about the system.*

B – Now, that you have convinced me to use the graph calculus, tell me how can I prove that it solves the original question of mine: deciding the valid inclusions between positive relational terms.
A – I'm afraid I won't have enough time to discuss it with you, giving all the details, as I'm on my way to the Agora. But, I think I can give you the main ideas.
B – I already know that one can associate to each term R a graph, namely the graph $G_R := (\{x,y\},\{xRy\},x,y)$, which defines the same relation as the term.

4.4 Soundness and completeness

A – We can show that $\models R \sqsubseteq S$, whenever $G_R \vdash G_S$, and vice-versa.

Proposition 4.5 (Soundness of +RG)
If $G_R \vdash G_S$, then $\models R \sqsubseteq S$.

B – Soundness seems easy to be established, but proving completeness looks like a tough task.

THEOREM 4.6 (Completeness of $+\mathsf{RG}$)
If $\models R \sqsubseteq S$ then $G_R \vdash G_S$.

A – I will give you some directions on how to establish this result. As we discussed before, the main idea behind it is to define a normal form for graphs and use it to prove that $\models R \sqsubseteq S$ by executing the following two major steps. First, reduce the graphs G_R and G_S to their normal forms. Second, verify whether or not NFG_S can be obtained from NFG_R by a single application of the homomorphism rule.
B – The first step (reduction to simple normal form) seems quite clear.
A – The second step can be established by constructing a (finite) canonical model. Given a slice $S = (N, A, x, y)$ in NF, its *canonical model* is $\mathfrak{C}_S := \langle N, (r^{\mathfrak{C}_S})_{r \in \mathsf{Rvar}} \rangle$, where $r^{\mathfrak{C}_S} := \{(u, v) \in N \times N : urv \in A\}$.

PROPOSITION 4.7 (Inclusion and homomorphism)
Given graphs G and H in NF, the following are equivalent.
(a) $\models G \sqsubseteq H$;
(b) for each slice S of G, its distinguished pair (x_S, y_S) is in $[\![H]\!]_{\mathfrak{C}_S}$;
(c) G is homomorphic to H.

PROOF. (a)⇒(b) is clear, as $(x_S, y_S) \in [\![G]\!]_{\mathfrak{C}_S}$. (b)⇒(c): for some slice T of H, $(x_S, y_S) \in [\![T]\!]_{\mathfrak{C}_S}$, so we have an assignment $g : T \to \mathfrak{C}_S$, which gives a slice homomorphism from T to S. (c)⇒(a) is clear, as rule GrHom is sound. ∎

A – By applying this Proposition 4.7 to graphs of terms, we immediately get completeness of $+\mathsf{RG}$.
B – This is quite interesting. Thus, to obtain the completeness result one just needs to show that your rules can execute the two major steps described.
A – Yes, and note that, as a bonus of this result, we obtain a standard form for the proofs of the logically valid inferences of $+\mathsf{RG}$.
B – Indeed, I can see that one has normal derivations in this graph calculus. Given G_R and G_S such that $G_R \models G_S$, a normal derivation of G_S from G_R has the form:

$$\begin{array}{l} G_R \\ \Downarrow \ \textit{elimination} \\ \mathrm{NF}G_R \\ \uparrow \ \textit{homomorphism} \\ \mathrm{NF}G_S \\ \Downarrow \ \textit{introduction} \\ G_S \end{array}$$

I can also see that the decidability of the validity problem for inclusions follows from the normal form property. Nevertheless I prefer the form of the *easy proofs* obtained by the use of the slice structural rules:

$$\begin{array}{l} G_R \\ \Downarrow \ \textit{elimination} \\ \mathrm{NF}G_R \\ \downarrow \ (\mathsf{SpltNd}^*; \mathsf{EraArc}^*; \mathsf{EraNd}^*)^* \\ \downarrow \ \mathsf{AddSl}^* \\ \mathrm{NF}G_S \\ \Uparrow \ \textit{elimination} \\ G_S \end{array}$$

A – Other results, as finite model property, follow from the normal proof property or, if you prefer, from the previous Proposition 4.7 on inclusion and homomorphism. Unfortunately, I don't have more time to talk with you about this matters today. Let's meet again later to start a joint work on the graph calculus.

5 Ongoing Work

In fact, after these meetings, Asinus and Buridanus started working together on the graph calculus. They are proving the technical details of the results mentioned in Section 4 as well as some other results. They hope to increase their understanding of this calculus and to extend it.

A – One of the things we should do right way is clarifying the connections between graphs and first-order formulas: that one one may regard graphs as equivalence classes of such formulas. I think we can first establish that the expressive power of our graphs is that of the positive existential first order language of binary relations having at most two free (fixed) variables.
B – Yes, the intuition seems quite clear to me. For instance, terms $R \sqcap S$ and $S \sqcap R$ are different, as are formulas $x(R \sqcap S)y$ and $x(S \sqcap R)y$. Yet, both are represented by the same graph with two parallel arcs xRy and xSy. Of course, we have to spell out the details. I imagine that, following your idea on the expressive power, we can correlate graphs with such positive existential formulas in disjunctive normal form.
A – In fact, I think that this result will establish a relationship between the graph rules and the first-order introduction and elimination rules, which may help to clear your doubts once and for all.
B – Also, it seems easy enough to extend the graph language and calculus to cope with the empty relation. It can be represented by the graph with no slices, for the meaning of such a graph is exactly $\emptyset$.
A – Indeed, and the rules would be quite simple too. In addition to the obvious ones for Boolean operators, we add rules stating that $\emptyset^{\mathsf{T}} = \emptyset$ and $R \circ \emptyset = \emptyset = \emptyset \circ R$. Thus, we may be able to eliminate the constant for $\emptyset$ from any graph containing it or reducing it to this constant.
B – Of course, complementation is another matter. If we add it too, we will have a graph calculus for the full relational terms. We are bound to lose some properties, such as decidability. Perhaps, we should start with a limited form of complementation: only of relation variables.
A – On the other hand, it would be nice to extend our graph calculus to derive inclusions from other inclusions taken as hypotheses: for consequences of the form $R_1 \sqsubseteq S_1, \ldots, R_n \sqsubseteq S_n \models R \sqsubseteq S$. If we do manage to do this, we could perhaps introduce the complement by definition. Then, this would mean that derivation from hypotheses will cease to be decidable. But I guess we can't eat our cake and have it.
B – Yes, there are many interesting problems ahead waiting for us.
A – I feel just like a kid playing on the seashore of an immense ocean of wonders ...

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